The $\mathrm{FP}^{\mathrm{NP}}$ versus #P Dichotomy for #EO

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April 23rd, 2025

1. Introduction
2. Known results
3. Our results
4. Open problems

Counting problems

Examples:

- Combinatorics: Counting the number of matchings, perfect matchings, vertex covers, etc.
- Statistic physics: Partition function.
- Counting feasible solutions of SAT problems.
- Counting the number of Eulerian orientations.

Counting problems

	Decision	Counting
Target	Is there a solution?	How many are solutions?
Complexity class	NP	#P

The classification theorem (usually dichotomy): given a problem, decide whether it's tractable (usually in P) or computationally hard.

Why can we use $\mathsf{FP}^{\mathsf{NP}}$?
If $\#\mathsf{P} \subseteq \mathsf{FP}^{\mathsf{NP}}$, then we have

$$\mathsf{PH} \subseteq \mathsf{FP}^{\#\mathsf{P}} \subseteq \mathsf{FP}^{\mathsf{FP}^{\mathsf{NP}}} = \mathsf{FP}^{\mathsf{NP}} = \Delta_2^{\mathsf{P}} \subseteq \mathsf{PH}.$$

Frameworks

Frameworks of counting problems: Spin system, #CSP, Holant*, Holant^c, #EO, etc.

#EO is no less expressive than #CSP, given the dichotomy of #CSP.

Counting Eulerian orientations

G is an undirected Eulerian graph, with vertices having even degree.

An **Eulerian Orientation** of G is an orientation of its edges such that each vertex's in-degree and out-degree are the same.

Counting the number of Eulerian orientations on undirected Eulerian graphs is #P-complete[1].

#EO problems[2]

A generalization: counting weighted Eulerian orientations (#EO).

Formalize the problem to add constraint signatures.

We consider complex weighted Boolean signatures in this work.

Definitions[2]

supp$(f) = \{\alpha \mid f(\alpha) \neq 0\}$, wt$(\alpha)$ is the number of 1's in α.

EO signature

An EO signature's support only contains half Hamming weight strings.
$\forall \alpha \in$ supp(f), wt(α) =arity$(f)/2$.

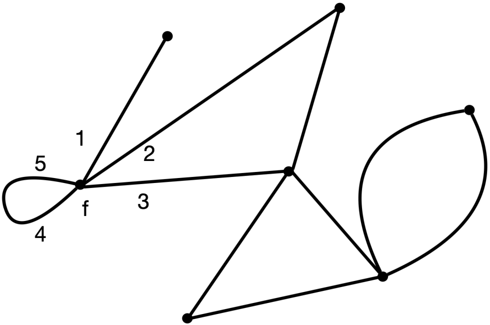
Signature grid

Given a set of EO signatures $\mathcal{F}$.

An EO signature grid $\Omega = (G, \pi)$ over $\mathcal{F}$ consists of a Eulerian graph $G = (V, E)$ and a mapping π that assigns to each vertex $v \in V$ an $f_v \in \mathcal{F}$ and a linear order of the incident edges at v.

General signature grid

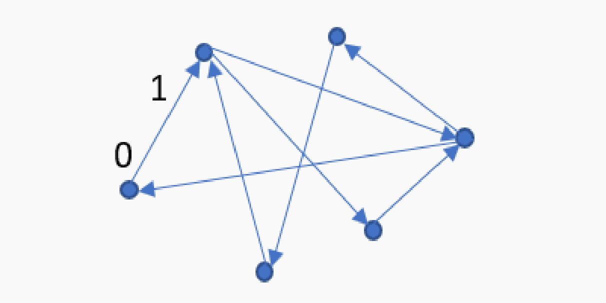
Here is an example of a general signature grid.



We further require the underlined graph to be Eulerian.

Definitions[2]

EO(G): the set of all Eulerian Orientations of G.
Each edge has two ends. Given a directed edge, assign 1 to the head and 0 to the tail.
The numbers of 0's and 1's at each v are equal.



#EO problems[2]

#EO problems

A #EO problem #EO($\mathcal{F}$) specified by a set $\mathcal{F}$ of EO signatures is the following: The input is an EO signature grid $\Omega = (G, \pi)$ over $\mathcal{F}$; the output is the partition function of Ω,

$$\#\mathrm{EO}_\Omega = \sum_{\sigma \in \mathrm{EO}(G)} \prod_{v \in V} f_v(\sigma|_{E(v)}).$$

Motivations

- Dichotomy theorems for counting problems are important and interesting.
- The dichotomy for #EO is a cornerstone of the classification theorem for complex Holant, which is one of the most general theorems.
- #EO itself is no less expressive than #CSP, given the dichotomy of #CSP.

To some extent, solving #EO is inevitable.

Known results

There are already some explorations in this problem.

Six vertex model[3]

A dichotomy for $\#\mathrm{EO}(f)$ problems, where f is an EO signature with arity 4.

Cai, Fu, Xia: Complexity classification of the six-vertex model. Information and Computation. 2018

Known results

EO signatures with ARS[2]

Let $\mathcal{F}$ be a set of EO signatures satisfying ARS. Then, #EO($\mathcal{F}$) is #P-hard unless $\mathcal{F} \subseteq \mathscr{A}$ or $\mathcal{F} \subseteq \mathscr{P}$, in which cases it is tractable. A EO signature f satisfies ARS iff for all α, $f(\overline{\alpha}) = \overline{f(\alpha)}$.

Cai, Fu, Shao: Beyond #CSP: A dichotomy for counting weighted Eulerian orientations with ARS. Information and Computation. 2020

This result contributes to the classification of Real Holant problems[4].

1. Introduction
2. Known results
3. **Our results**
   - Reduction from $\#\mathrm{EO}^c$ to $\#\mathrm{EO}$
   - Key challenge: subtle supports
   - A dichotomy for $\#\mathrm{EO}$
4. Open problems

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Matrix form

We often use the matrix form to explicitly denote a signature.

$x_1x_2 \backslash x_3x_4$	00	01	10	11
00	0	0	0	a
01	0	b	0	0
10	0	c	0	0
11	0	0	0	0

Such matrix represents a signature with support $\{0011, 0101, 0110\}$.

Reduction from $\#\mathsf{EO}^c$ to $\#\mathsf{EO}$

$$\#\mathsf{EO}^c(\mathcal{F}) = \#\mathsf{EO}(\mathcal{F} \bigcup \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}).$$

The auxiliary signature $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is significant for reducing arity and further analysis.

We prove that under certain conditions, $\#\mathsf{EO}^c(\mathcal{F}) \leq_T \#\mathsf{EO}(\mathcal{F})$.

Why supports matter?

Affine supports usually imply the tractability to some extent. (Solving linear equations is polynomial-time computable).

With further assumptions towards values, we get $\mathscr{A}$ and $\mathscr{P}$, which are all tractable cases for Boolean #CSP.

In the setting of symmetric Holant problems, vanishing signatures imply more structures of tractable signatures' supports.

In the setting of #EO, the "tractable" structure is more complicated.

Pairwise opposite[2]

Pairwise opposite

A 2n-arity signature f is called pairwise opposite iff its variables can be divided into n pairs. And in each pair the two variables are always opposite.

All affine EO signatures are pairwise opposite.

If all signatures are pairwise opposite, the #EO problem can be transformed to a #CSP problem to analyze its computational complexity.

Affine span

3-span

3-span(f)=$\{\alpha \oplus \beta \oplus \gamma \mid \alpha, \beta, \gamma \in \text{supp}(f)\}$.

$(2k+1)$-span(f) can be similarly defined.

Why do we define these sets? The smallest affine space that contains supp(f) is $\bigcup_{k=1}^{\infty}$(2k+1)-span(f).

Affine span

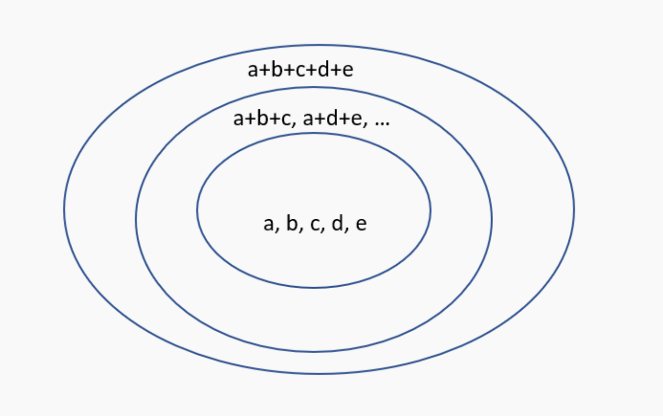


图 1: Example of 3-span and 5-span

Affine span

3-span is surprisingly more important than others!

A theorem for hardness

Suppose f is a 2n-arity signature. If there is an $\alpha \in$3-span$(f) - \mathrm{supp}(f)$ with $\mathrm{wt}(\alpha) = n$, then $\#\mathrm{EO}(f)$ is #P-hard.

Affine span

With the tensor operation, we can easily get a corollary:

Corollary

Suppose f is a signature of arity $2n$. If there are $\alpha, \beta \in$3-span$(f) -$ supp(f) with wt$(\alpha) < n$ and wt$(\beta) > n$, then #EO(f) is #P-hard.

We now focus on signatures with a pure 3-span, i.e., 3-span$(f) -$ supp(f) only contains 01-strings with Hamming weight more (or less) than $\frac{1}{2}$arity(f).

A $\mathsf{FP}^{\mathsf{NP}}$ algorithm

We consider signatures with pure-up 3-span.
$\forall \alpha, \beta, \gamma \in \text{supp}(f)$, $\alpha \oplus \beta \oplus \gamma$ has more 1's than 0's.

The closure property

Suppose $\mathcal{F}$ contains signatures with pure-up 3-span, then any $\mathcal{F}$-gate has pure-up 3-span.

A $\mathrm{FP}^{\mathrm{NP}}$ algorithm

Using the closure property and some analysis, we have:

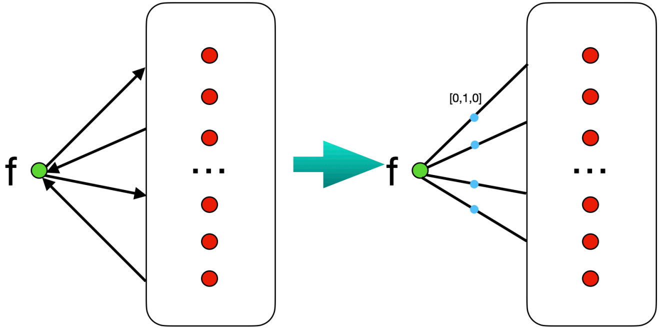
Theorem

Given an instance I of $\#\mathrm{EO}(\mathcal{F})$, where $\mathcal{F}$ contains signatures with pure-up 3-span, then for each vertex, only part of the support of f_v has contribution to $\mathrm{Z}(I)$, where $\mathrm{Z}(I)$ is the partition function of I. The effective part of the support of each vertex is pairwise opposite.

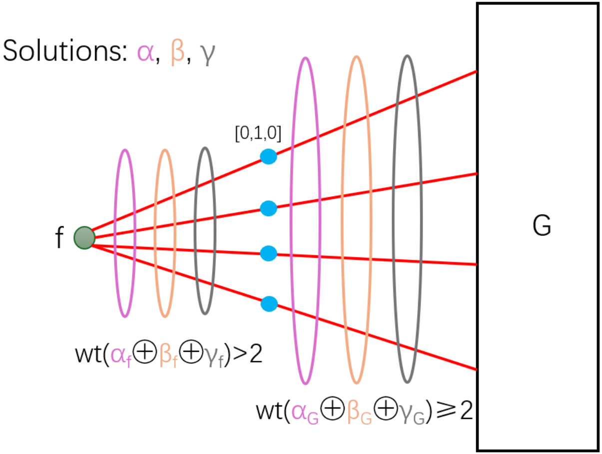
We focus on effective supports!

A $\mathrm{FP}^{\mathrm{NP}}$ algorithm

Suppose arity(f)=4, $\alpha_f, \beta_f, \gamma_f \in$ supp(f) and wt($\alpha_f \oplus \beta_f \oplus \gamma_f$) > 2, then at most two of them are effective.



A FP^{NP} algorithm



A contradiction!

A $\mathrm{FP}^{\mathrm{NP}}$ algorithm

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \color{red}{b} & 0 & 0 \\ 0 & \color{red}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only b and c are effective in the partition function. Therefore, f_v can be seen equivalently in this instance as:

$$\begin{pmatrix} 0 & 0 & 0 & \color{blue}{0} \\ 0 & \color{red}{b} & 0 & 0 \\ 0 & \color{red}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is in $\mathscr{P}$.

A $\mathrm{FP}^{\mathrm{NP}}$ algorithm

Using this observation, we can present a $\mathrm{FP}^{\mathrm{NP}}$ algorithm:

$\mathrm{FP}^{\mathrm{NP}}$ algorithm

Given an problem $\#\mathrm{EO}(\mathcal{F})$, where $\mathcal{F}$ contains signatures with pure-up 3-span. Given an NP-oracle, we can decide the effective supports of all signatures in an instance and whether $\#\mathrm{EO}(\mathcal{F})$ is $\#P$-hard or not. The whole process is in polynomial time.

Main theorem

Main theorem

$\mathcal{F}$ is a set of #EO signatures, then #EO is either in $\mathrm{FP}^{\mathrm{NP}}$ or #P-hard. The classification criteria are explicit.

The work is presented in Meng, Wang, Xia, The $\mathrm{FP}^{\mathrm{NP}}$ versus #P Dichotomy for #EO, STOC'25 [5].

A trichotomy?

Given a problem $\#\mathrm{EO}(\mathcal{F})$, we know there are only three situations:

- $\#$P-hard;
- In $\mathrm{FP}^{\mathrm{NP}}$;
- In FP.

The first two situations completely classified $\#$EO. However, whether a problem in $\mathrm{FP}^{\mathrm{NP}}$ is in FP or not is still unknown.

Is the corresponding oracle NP-complete?

f_{56}

f_{56} is a very sparse 01-value signature. Its arity is 56, and its support only contains five 01-strings.

	*4										*3				
0	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0
0	1	0	0	0	1	1	1	0	0	0	1	1	1	0	1
0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
0	0	0	1	0	0	1	0	1	0	1	1	0	1	1	1
0	0	0	0	1	0	0	1	0	1	1	0	1	1	1	1

f_{56}

It's 3-span is pure-up, but the sum of five distinct string is $\mathbf{0}$.

We know it's in $\mathrm{FP}^{\mathrm{NP}}$. Suppose $\mathrm{FP}^{\mathrm{NP}} \neq \#\mathrm{P}$, it's definitely not #P-hard, but we don't know whether $\#\mathrm{EO}(f_{56})$ is in FP or NP-complete.

If it's NP-complete, then #EO problems can be surprisingly classified to three cases, which is a situation we haven't seen before.

Open problems

Unfortunately, the complexity of Edge-CSP$(\neq_2 | f_{56})$ is not yet classified.

All signatures that have pure-up(down) 3-span but don't have P_0 and P_1 property are subtle. We know the problems are in $\mathsf{FP}^{\mathsf{NP}}$, but can't go further.

[1] M. Mihail and P. Winkler, "On the number of eulerian orientations of a graph," *Algorithmica*, vol. 16, pp. 402–414, 1996.

[2] J.-Y. Cai, Z. Fu, and S. Shao, "Beyond# csp: A dichotomy for counting weighted eulerian orientations with ars," *Information and Computation*, vol. 275, p. 104589, 2020.

[3] J.-Y. Cai, Z. Fu, and M. Xia, "Complexity classification of the six-vertex model," *Information and Computation*, vol. 259, pp. 130–141, 2018.

[4] S. Shao and J.-Y. Cai, "A dichotomy for real boolean holant problems," in *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2020, pp. 1091–1102.

[5] B. Meng, J. Wang, and M. Xia, "The FP^{NP} versus #P dichotomy for #EO," *arXiv preprint arXiv:2502.02012*, 2025.

Thanks!